1 **a** $x^2 + y^2 = 25$ **b** $(x-1)^2 + (y-3)^2 = 4$ **c** $(x-4)^2 + (y+6)^2 = 1$

d $(x+1)^2 + (y+8)^2 = 9$ **e** $(x+\frac{1}{2})^2 + (y-\frac{1}{2})^2 = \frac{1}{4}$ **f** $(x+3)^2 + (y-9)^2 = 12$

2 **a** centre (0, 0) radius 4 **b** centre (6, 1) radius 9 **c** centre (–1, 4) radius 11

d centre (7, 0) radius 0.3 **e** centre (–2, –5) radius $4\sqrt{2}$ **f** centre (8, –9) radius $6\sqrt{3}$

3 **a** $x^2 + (y-2)^2 - 4 + 3 = 0$
$x^2 + (y-2)^2 = 1$
centre (0, 2) radius 1

b $(x-1)^2 - 1 + (y-5)^2 - 25 - 23 = 0$
$(x-1)^2 + (y-5)^2 = 49$
centre (1, 5) radius 7

c $(x+6)^2 - 36 + (y-4)^2 - 16 + 36 = 0$
$(x+6)^2 + (y-4)^2 = 16$
centre (–6, 4) radius 4

d $(x-1)^2 - 1 + (y+8)^2 - 64 = 35$
$(x-1)^2 + (y+8)^2 = 100$
centre (1, –8) radius 10

e $(x-4)^2 - 16 + (y+3)^2 - 9 = 0$
$(x-4)^2 + (y+3)^2 = 25$
centre (4, –3) radius 5

f $(x+5)^2 - 25 + (y-1)^2 - 1 - 19 = 0$
$(x+5)^2 + (y-1)^2 = 45$
centre (–5, 1) radius $3\sqrt{5}$

g $x^2 + y^2 - x - 6y + \frac{1}{4} = 0$
$(x-\frac{1}{2})^2 - \frac{1}{4} + (y-3)^2 - 9 + \frac{1}{4} = 0$
$(x-\frac{1}{2})^2 + (y-3)^2 = 9$
centre ($\frac{1}{2}$, 3) radius 3

h $x^2 + y^2 + \frac{2}{3}x - \frac{8}{3}y + \frac{8}{9} = 0$
$(x+\frac{1}{3})^2 - \frac{1}{9} + (y-\frac{4}{3})^2 - \frac{16}{9} + \frac{8}{9} = 0$
$(x+\frac{1}{3})^2 + (y-\frac{4}{3})^2 = 1$
centre ($-\frac{1}{3}$, $\frac{4}{3}$) radius 1

4 **a** radius = $\sqrt{9+16} = 5$ $\therefore$ $(x-1)^2 + (y+2)^2 = 25$

b radius = $\sqrt{25+4} = \sqrt{29}$ $\therefore$ $(x+5)^2 + (y-7)^2 = 29$

5 **a** centre ($\frac{1+3}{2}$, –2) = (2, –2)
radius = 1
$\therefore (x-2)^2 + (y+2)^2 = 1$

b centre ($\frac{-7+1}{2}$, $\frac{2+8}{2}$) = (–3, 5)
radius = $\sqrt{16+9} = 5$
$\therefore (x+3)^2 + (y-5)^2 = 25$

c centre ($\frac{1+4}{2}$, $\frac{1+0}{2}$) = ($\frac{5}{2}$, $\frac{1}{2}$)
radius = $\sqrt{\frac{9}{4}+\frac{1}{4}} = \sqrt{\frac{5}{2}}$
$\therefore (x-\frac{5}{2})^2 + (y-\frac{1}{2})^2 = \frac{5}{2}$

6 **a** grad $PQ = \frac{10-1}{3-0} = 3$, grad $QR = \frac{9-10}{6-3} = -\frac{1}{3}$
grad $PQ \times$ grad $QR = 3 \times (-\frac{1}{3}) = -1$
$\therefore$ *PQ* and *QR* are perpendicular
$\therefore$ $\angle PQR$ is a right-angle

b $\angle PQR$ is a right-angle $\therefore$ *PR* is a diameter of *C*
$\therefore$ centre is ($\frac{0+6}{2}$, $\frac{1+9}{2}$) = (3, 5)
radius = 5
$\therefore$ $(x-3)^2 + (y-5)^2 = 25$
$x^2 - 6x + 9 + y^2 - 10y + 25 - 25 = 0$
$x^2 + y^2 - 6x - 10y + 9 = 0$

7 **a** centre (0, 0) radius 8
dist. pt to centre = 9
$\therefore$ outside circle

b $(x-1)^2 - 1 + (y-3)^2 - 9 - 26 = 0$
$(x-1)^2 + (y-3)^2 = 36$
centre (1, 3) radius 6
dist. pt to centre = $\sqrt{9+16} = 5$
$\therefore$ inside circle

c $(x+5)^2 - 25 + (y-2)^2 - 4 = 140$
$(x+5)^2 + (y-2)^2 = 169$
centre (–5, 2) radius 13
dist. pt to centre = $\sqrt{144+25} = 13$
$\therefore$ on circle

d $(x+1)^2 - 1 + (y+4)^2 - 16 - 13 = 0$
$(x+1)^2 + (y+4)^2 = 30$
centre (–1, –4) radius $\sqrt{30}$
dist. pt to centre = $\sqrt{9+25} = \sqrt{34}$
$\therefore$ outside circle

8 $(x+6)^2 - 36 + (y-3)^2 - 9 + 27 = 0$
$(x+6)^2 + (y-3)^2 = 18$
centre (–6, 3) radius $3\sqrt{2}$
dist. Q to centre = $\sqrt{196+4} = 10\sqrt{2}$
min. $PQ = 10\sqrt{2} - 3\sqrt{2} = 7\sqrt{2}$

9 x-coord of centre = $\frac{2+8}{2} = 5$
y-coord of centre = 4 $\therefore$ centre (5, 4)
radius = dist. (0, 4) to (5, 4) = 5
$\therefore$ $(x-5)^2 + (y-4)^2 = 25$

10 $(x+4)^2 - 16 + (y-6)^2 - 36 + k = 0$
$(x+4)^2 + (y-6)^2 = 52 - k$
centre (–4, 6) $r^2 = 52 - k$
$r > 0$ $\therefore$ $k < 52$
also require $r < 4$
$\therefore$ $52 - k < 16$
$k > 36$
$\therefore$ $36 < k < 52$

11 **a** mid-point $PQ = (\frac{-2+2}{2}, \frac{-2+(-4)}{2}) = (0, -3)$
grad $PQ = \frac{-4+2}{2+2} = -\frac{1}{2}$
perp. grad = 2
$\therefore$ $y = 2x - 3$

b mid-point $PR = (\frac{-2+7}{2}, \frac{-2+1}{2}) = (\frac{5}{2}, -\frac{1}{2})$
grad $PR = \frac{1+2}{7+2} = \frac{1}{3}$
perp. grad = –3
perp. bisector $y + \frac{1}{2} = -3(x - \frac{5}{2})$
$y = 7 - 3x$
centre where intersect $2x - 3 = 7 - 3x$
$x = 2$ $\therefore$ (2, 1)

c radius = dist. (2, 1) to (7, 1) = 5
$\therefore$ $(x-2)^2 + (y-1)^2 = 25$

12 **a** $(x-2)^2 - 4 + (y-2)^2 - 4 - 28 = 0$
$(x-2)^2 + (y-2)^2 = 36$
centre (2, 2) radius 6
dist. = $\sqrt{64+36} = 10$

b tangent perp to radius
$\therefore$ $AB^2 = 10^2 - 6^2 = 64$
$AB = 8$

13 $(x+3)^2 - 9 + (y-1)^2 - 1 = 0$
$(x+3)^2 + (y-1)^2 = 10$
centre (–3, 1) radius $\sqrt{10}$
dist. centre to (2, 6) = $\sqrt{25+25} = \sqrt{50}$
$PQ^2 = (\sqrt{50})^2 - (\sqrt{10})^2 = 40$
$PQ = \sqrt{40} = 2\sqrt{10}$

14 **a** $(x-3)^2 - 9 + (y-5)^2 - 25 + 16 = 0$
$\therefore$ centre $(3, 5)$
b grad $= \frac{5-2}{3-6} = -1$
c $y - 2 = -(x-6)$ $\quad [\, y = 8 - x \,]$

15 **a** $(x+2)^2 - 4 + y^2 = 13$
$\therefore$ centre $(-2, 0)$
grad $= \frac{0-4}{-2+1} = 4$
$\therefore$ $y - 4 = 4(x+1)$ $\quad [\, y = 4x + 8 \,]$
b $(x+1)^2 - 1 + (y+2)^2 - 4 - 40 = 0$
$\therefore$ centre $(-1, -2)$
grad normal $= \frac{-2-1}{-1-5} = \frac{1}{2}$
$\therefore$ grad tangent $= -2$
$\therefore$ $y - 1 = -2(x-5)$ $\quad [\, y = 11 - 2x \,]$
c $(x-5)^2 - 25 + (y+2)^2 - 4 + 4 = 0$
$\therefore$ centre $(5, -2)$
grad normal $= \frac{-2-2}{5-2} = -\frac{4}{3}$
$\therefore$ grad tangent $= \frac{3}{4}$
$\therefore$ $y - 2 = \frac{3}{4}(x-2)$ $\quad [\, 3x - 4y + 2 = 0 \,]$

16 $x = 0 \Rightarrow y^2 + 6y - 16 = 0$
$(y+8)(y-2) = 0$
$y = -8, 2$
$y = 0 \Rightarrow x^2 - 6x - 16 = 0$
$(x+2)(x-8) = 0$
$x = -2, 8$
$\therefore$ $(0, -8)$, $(0, 2)$, $(-2, 0)$ and $(8, 0)$

17 **a** sub. $x^2 + (x-4)^2 = 10$
$x^2 - 4x + 3 = 0$
$(x-1)(x-3) = 0$
$x = 1, 3$
$\therefore$ $(1, -3)$ and $(3, -1)$
b sub. $y = 17 - 3x$
$x^2 + (17-3x)^2 - 4x - 2(17-3x) - 15 = 0$
$x^2 - 10x + 24 = 0$
$(x-4)(x-6) = 0$
$x = 4, 6$
$\therefore$ $(4, 5)$ and $(6, -1)$
c sub.
$4x^2 + 4(2x+2)^2 + 4x - 8(2x+2) - 15 = 0$
$4x^2 + 4x - 3 = 0$
$(2x+3)(2x-1) = 0$
$x = -\frac{3}{2}, \frac{1}{2}$
$\therefore$ $(-\frac{3}{2}, -1)$ and $(\frac{1}{2}, 3)$

18 sub.
$x^2 + (1-x)^2 + 6x + 2(1-x) = 27$
$x^2 + x - 12 = 0$
$(x+4)(x-3) = 0$
$x = -4, 3$
$\therefore$ $(-4, 5)$ and $(3, -2)$
$AB = \sqrt{49+49} = 7\sqrt{2}$

19 sub.
$x^2 + (2x+1)^2 - 8x - 8(2x+1) + 27 = 0$
$x^2 - 4x + 4 = 0$
$(x-2)^2 = 0$
repeated root $\therefore$ tangent
touch when $x = 2$ $\therefore$ at $(2, 5)$

20 sub.
$x^2 + (x + k)^2 + 6x - 8(x + k) + 17 = 0$
$2x^2 + (2k - 2)x + k^2 - 8k + 17 = 0$
tangent $\therefore$ repeated root $\therefore$ $b^2 - 4ac = 0$
$\Rightarrow$ $(2k - 2)^2 - 8(k^2 - 8k + 17) = 0$
$k^2 - 14k + 33 = 0$
$(k - 3)(k - 11) = 0$
$\therefore$ $k = 3$ or 11

21 sub.
$x^2 + m^2x^2 - 8x - 16mx + 72 = 0$
$(1 + m^2)x^2 - (8 + 16m)x + 72 = 0$
tangent $\therefore$ repeated root $\therefore$ $b^2 - 4ac = 0$
$\Rightarrow$ $(8 + 16m)^2 - 288(1 + m^2) = 0$
$m^2 - 8m + 7 = 0$
$(m - 1)(m - 7) = 0$
$\therefore$ $m = 1, 7$

22 sub. $x = \frac{k-3y}{2}$
$(\frac{k-3y}{2})^2 + y^2 + 6(\frac{k-3y}{2}) + 4y = 0$
$(k - 3y)^2 + 4y^2 + 12(k - 3y) + 16y = 0$
$13y^2 - (6k + 20)y + k^2 + 12k = 0$
tangent $\therefore$ repeated root $\therefore$ $b^2 - 4ac = 0$
$\Rightarrow$ $(6k + 20)^2 - 52(k^2 + 12k) = 0$
$k^2 + 24k - 25 = 0$
$(k + 25)(k - 1) = 0$
$\therefore$ $k = -25, 1$

23 **a** $x = 0 \Rightarrow y^2 - 6y - 7 = 0$
$(y + 1)(y - 7) = 0$
$y = -1, 7$
$\therefore$ $(0, -1)$ and $(0, 7)$

b $(x - 2)^2 - 4 + (y - 3)^2 - 9 = 7$
$\therefore$ centre $(2, 3)$
grad normal at $(0, -1) = \frac{3+1}{2-0} = 2$
$\therefore$ grad tangent at $(0, -1) = -\frac{1}{2}$
$\therefore$ $y = -\frac{1}{2}x - 1$
grad normal at $(0, 7) = \frac{3-7}{2-0} = -2$
$\therefore$ grad tangent at $(0, 7) = \frac{1}{2}$
$\therefore$ $y = \frac{1}{2}x + 7$
intersect when $-\frac{1}{2}x - 1 = \frac{1}{2}x + 7$
$x = -8$
$\therefore$ $(-8, 3)$

1 **a** $(x-3)^2 + (y+2)^2 = 25$

b sub. $(x-3)^2 + [(2x-3)+2]^2 = 25$

$(x-3)^2 + (2x-1)^2 = 25$

$x^2 - 2x - 3 = 0$

$(x+1)(x-3) = 0$

$x = -1, 3$

$\therefore$ $(-1, -5)$ and $(3, 3)$

$AB^2 = 4^2 + 8^2 = 80$

$AB = \sqrt{80} = \sqrt{16\times5} = 4\sqrt{5}$

2 **a** $= (\frac{-5+3}{2}, \frac{6+8}{2}) = (-1, 7)$

b radius $= \sqrt{16+1} = \sqrt{17}$

$\therefore$ $(x+1)^2 + (y-7)^2 = 17$

c grad of radius $= \frac{7-6}{-1-(-5)} = \frac{1}{4}$

$\therefore$ grad of tangent $= -4$

$\therefore$ $y - 6 = -4(x+5)$

$[\, y = -4x - 14 \,]$

3 **a** $(x+4)^2 - 16 + (y-8)^2 - 64 + 62 = 0$

$(x+4)^2 + (y-8)^2 = 18$

$\therefore$ centre $(-4, 8)$ radius $3\sqrt{2}$

b grad of $l = 2$ $\therefore$ grad of perp. $= -\frac{1}{2}$

eqn. of line perp to l through centre:

$y - 8 = -\frac{1}{2}(x+4)$

$y = 6 - \frac{1}{2}x$

intersects l when:

$2x + 1 = 6 - \frac{1}{2}x$

$x = 2$ $\therefore$ $(2, 5)$ is closest point

dist. $(2, 5)$ to centre

$= \sqrt{36+9} = \sqrt{45} = 3\sqrt{5}$

min. dist. $= 3\sqrt{5} - 3\sqrt{2} = 3(\sqrt{5} - \sqrt{2})$

4 **a** $PQ = \sqrt{1+9} = \sqrt{10}$

radius $= \frac{1}{2}PQ = \frac{1}{2}\sqrt{10}$

b = midpoint of PR

$= (\frac{0+7}{2}, \frac{4+3}{2}) = (\frac{7}{2}, \frac{7}{2})$

c midpoint of $PQ = (\frac{0+1}{2}, \frac{4+1}{2}) = (\frac{1}{2}, \frac{5}{2})$

centre of C_1 = midpoint of $(\frac{1}{2}, \frac{5}{2})$ and $(\frac{7}{2}, \frac{7}{2})$

$= (\frac{\frac{1}{2}+\frac{7}{2}}{2}, \frac{\frac{5}{2}+\frac{7}{2}}{2}) = (2, 3)$

$\therefore$ eqn. of C_1:

$(x-2)^2 + (y-3)^2 = (\frac{1}{2}\sqrt{10}\,)^2$

$x^2 - 4x + 4 + y^2 - 6y + 9 = \frac{5}{2}$

$2x^2 - 8x + 8 + 2y^2 - 12y + 18 = 5$

$2x^2 + 2y^2 - 8x - 12y + 21 = 0$

5 **a** midpoint $AB = (\frac{0+2}{2}, \frac{3+7}{2}) = (1, 5)$

grad $AB = \frac{7-3}{2-0} = 2$

$\therefore$ perp. grad $= -\frac{1}{2}$

$\therefore$ $y - 5 = -\frac{1}{2}(x-1)$

$[\, y = \frac{11}{2} - \frac{1}{2}x \,]$

b circle touches y-axis at $(0, 3)$

$\therefore$ y-coord of centre = 3

sub. $3 = \frac{11}{2} - \frac{1}{2}x$

$x = 5$

$\therefore$ centre $(5, 3)$ radius 5

$\therefore$ $(x-5)^2 + (y-3)^2 = 25$

c grad of radius $= \frac{7-3}{2-5} = -\frac{4}{3}$

$\therefore$ grad of tangent $= \frac{3}{4}$

$\therefore$ $y - 7 = \frac{3}{4}(x-2)$

$4y - 28 = 3x - 6$

$3x - 4y + 22 = 0$

6 $AP^2 = (x+3)^2 + (y-4)^2$

$BP^2 = x^2 + (y+2)^2$

$AP = 2BP$ $\therefore$ $AP^2 = 4BP^2$

$\therefore$ $(x+3)^2 + (y-4)^2 = 4[x^2 + (y+2)^2]$

$x^2 + 6x + 9 + y^2 - 8y + 16 = 4x^2 + 4y^2 + 16y + 16$

$x^2 - 2x + y^2 + 8y - 3 = 0$

$(x-1)^2 - 1 + (y+4)^2 - 16 - 3 = 0$

$(x-1)^2 + (y+4)^2 = 20$

in form $(x-a)^2 + (y-b)^2 = r^2$ $\therefore$ circle

centre $(1, -4)$ radius $2\sqrt{5}$

7 **a** $= (\frac{-4+(-2)}{2}, \frac{9+(-5)}{2}) = (-3, 2)$

b radius $= \sqrt{1+49} = \sqrt{50}$

$\therefore\ (x+3)^2 + (y-2)^2 = 50$

c sub. (2, 7) into eqn of C:

$(2+3)^2 + (7-2)^2 = 50$

$25 + 25 = 50$

true $\therefore$ R lies on C

d 90°

PQ is a diameter

$\therefore\ \angle PRQ$ is the angle in a semicircle

8 **a** $x^2 + (y-2)^2 - 4 - 16 = 0$

$\therefore$ centre (0, 2)

b C_2: $(x-1)^2 - 1 + (y-4)^2 - 16 - 60 = 0$

$\therefore$ centre (1, 4)

grad $= \frac{4-2}{1-0} = 2$

$\therefore\ y = 2x + 2$

c sub. into eqn of C_1:

$x^2 + [(2x+2) - 2]^2 - 20 = 0$

$x^2 + (2x)^2 - 20 = 0$

$x^2 = 4$

$x = \pm 2$

from diagram, $x = -2$ at P

$\therefore\ P\ (-2, -2)$

l perp to line through centres

$\therefore$ grad $= -\frac{1}{2}$

$\therefore\ y + 2 = -\frac{1}{2}(x+2)$

$[\ y = -\frac{1}{2}x - 3\ ]$

9 **a** $(x-4)^2 - 16 + (y+2)^2 - 4 + 12 = 0$

$(x-4)^2 + (y+2)^2 = 8$

centre (4, –2) radius $2\sqrt{2}$

b dist. P to centre

$= \sqrt{1+49} = \sqrt{50} = 5\sqrt{2}$

$\therefore$ max. $PQ = 5\sqrt{2} + 2\sqrt{2} = 7\sqrt{2}$

min. $PQ = 5\sqrt{2} - 2\sqrt{2} = 3\sqrt{2}$

c tangent perp. to radius

$PQ^2 = (5\sqrt{2})^2 - (2\sqrt{2})^2 = 50 - 8 = 42$

$PQ = \sqrt{42} = 6.48$

10 **a** radius $= b$

$\therefore\ (x-a)^2 + (y-b)^2 = b^2$

b sub. $y = x$ into eqn

$(x-a)^2 + (x-b)^2 = b^2$

$x^2 - 2ax + a^2 + x^2 - 2bx + b^2 = b^2$

$2x^2 - 2(a+b)x + a^2 = 0$

tangent $\therefore$ repeated root

$\therefore$ "$b^2 - 4ac$" $= 0$

$4(a+b)^2 - 8a^2 = 0$

$a^2 - 2ab - b^2 = 0$

$a = \frac{2b \pm \sqrt{4b^2 + 4b^2}}{2} = b \pm \sqrt{2}\,b$

$a > 0, b > 0\ \therefore\ a = (1 + \sqrt{2})b$