

## PURE MATHEMATICS 1 PRACTICE A SOLUTIONS

1. Given that  $y = \frac{1}{16}x^4$ , express each of the following in the form  $kx^n$ , where  $k$  and  $n$

are constants.

(a)  $y^{\frac{1}{4}}$  (1)

(b)  $2y^{-1}$  (1)

(c)  $(4y)^{\frac{3}{2}}$  (1)

(Total 3 marks)

$$\begin{aligned} \text{a) } y^{\frac{1}{4}} &= \left(\frac{1}{16}x^4\right)^{\frac{1}{4}} \\ &= \left(\frac{1}{16}\right)^{\frac{1}{4}} \times (x^4)^{\frac{1}{4}} \\ &= \frac{1}{2} \times x \end{aligned}$$

$$= \frac{1}{2}x // \quad \text{(A1)}$$

$$\begin{aligned} \text{c) } (4y)^{\frac{3}{2}} &= \left(\frac{4}{16}x^4\right)^{\frac{3}{2}} \\ &= \left(\frac{1}{4}\right)^{\frac{3}{2}} \cdot (x^4)^{\frac{3}{2}} \\ &= \left(\frac{1}{4}\right)^{\frac{3}{2}} \cdot (x^4)^{\frac{3}{2}} \end{aligned}$$

$$= \frac{1}{8} \cdot x^6$$

$$\text{b, } 2y^{-1} = 2\left(\frac{1}{16}x^4\right)^{-1}$$

$$= \frac{x^6}{8} // \quad \text{(A1)}$$

$$= 2(16x^{-4})$$

$$= 32x^{-4}$$

$$= \frac{32}{x^4} // \quad \text{(A1)}$$

2. Find all the roots of the function  $f(x) = 4x - 14x^{\frac{1}{2}} + 6$

(Total 3 marks)

$$\begin{aligned} \text{let } y &= x^{\frac{1}{2}} \Rightarrow 14x^{\frac{1}{2}} = 14y \\ &\Rightarrow 4x = 4y^2 \end{aligned}$$

$$\therefore f(x) = 4y^2 - 14y + 6$$

$$\text{Solving when } f(x)=0 \Rightarrow 4y^2 - 14y + 6 = 0 \quad (\text{M1})$$

$$(y-3)(4y-2) = 0$$

$$y-3=0 \text{ or } 4y-2=0$$

$$y=3 \text{ or } y=\frac{1}{2} \quad (\text{A1})$$

$$(\text{Replacing } y \text{ with } x^{\frac{1}{2}}) \therefore x^{\frac{1}{2}} = 3 \quad x^{\frac{1}{2}} = \frac{1}{2}$$

$$(\text{Squaring on both sides}) \quad x = 9 \quad x = \frac{1}{4} \quad (\text{A1})$$

3. Find the set of values of  $k$  for which the equation  $kx^2 - (k+2)x + 3k = 2$

has no real roots, except for one value of  $k$  which must be stated. Give your answer in set notation and in exact form

(4)

(Total 4 marks)

a)  $kx^2 - (k+2)x + 3k - 2 = 0$

$$\begin{matrix} kx^2 & -k-2x & +3k-2 = 0 \\ \downarrow & \underbrace{\qquad}_{a} & \underbrace{\qquad}_{b} & \underbrace{\qquad}_{c} \end{matrix}$$

$$a = k, b = -k-2, c = 3k-2$$

Using  $b^2 - 4ac < 0$  (NO REAL ROOTS)

$$\therefore (-k-2)^2 - 4(k)(3k-2) < 0 \quad \text{(M1)}$$

$$k^2 + 4k + 4 - 12k^2 + 8k < 0$$

$$-11k^2 + 12k + 4 < 0$$

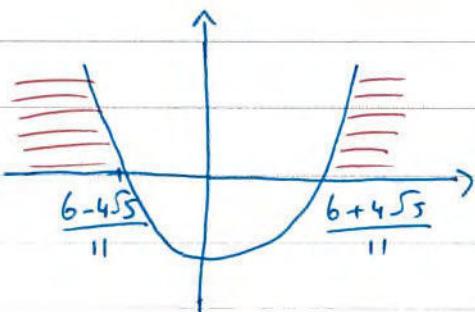
$$11k^2 - 12k - 4 > 0 \quad (\text{Multiply by } -1)$$

$$k = \frac{12 \pm \sqrt{(-12)^2 - 4(11)(-4)}}{2(11)}$$

$$= \frac{12 \pm \sqrt{320}}{22}$$

$$= \frac{12 \pm 8\sqrt{5}}{22}$$

$$= \frac{6 \pm 4\sqrt{5}}{11} \quad \text{(A1)}$$



$$\therefore \left. \begin{array}{l} k : k < \frac{6-4\sqrt{5}}{11} \quad \text{OR} \quad k > \frac{6+4\sqrt{5}}{11} \end{array} \right\} \quad \text{(A1)}$$

$b, k=0 //$  A1

The equation will always have NO REAL ROOTS unless  $k=0$  as it will turn the quadratic equation into a linear equation

$$\text{If } k=0 \Rightarrow 0x^2 - (0+2)x + 3(0) = 2$$
$$\downarrow$$
$$0 - 2x = 2$$

4. A ball is being projected from a point on the top of a building above the horizontal ground.  
The height, in metres, of the ball after  $t$  seconds can be modelled by the function:

$$h(t) = 13.7t + 7.8 - 2.9t^2$$

- (a) Use the model to find the height of the tower (1)
- (b) After how many seconds does the ball hit the ground (2)
- (c) Rewrite  $h(t)$  in the form  $A - B(t - C)^2$  where  $A, B$  and  $C$  are constants (3)
- (d) With reference to your answer in part (c) or otherwise, find the maximum height of the ball above the ground, and the time at which this maximum height is reached (2)

(Total 8 marks)

a) height of tower  $\Rightarrow$  initial time  $\Rightarrow t=0$

Replacing  $t=0$  in  $h(t) = 13.7t + 7.8 - 2.9t^2$   
 $\Rightarrow h(0) = 0 + 7.8 - 0$   
 $= 7.8$

$\therefore$  Height of tower = 7.8 // A1

b) When ball hits the ground  $\Rightarrow h(t) = 0$

$\therefore 13.7t + 7.8 - 2.9t^2 = 0$  M1

$\underbrace{2.9t^2}_{a} \underbrace{-13.7t}_{b} \underbrace{-7.8}_{c} = 0$  (Multiply by -1)

$a = 2.9, b = -13.7, c = -7.8$

Using  $t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \Rightarrow t = \frac{13.7 \pm \sqrt{(-13.7)^2 - 4(2.9)(-7.8)}}{2(2.9)}$

$t = \frac{13.7 \pm \sqrt{278.17}}{5.8}$

A1  $\therefore$  Time = 5.2 (2S.F) =  $t = 5.235$  or  $t = -0.513$  NOT VALID

$$c) h(t) = -2.9t^2 + 13.7t + 7.8$$

$$= -2.9 \left[ t^2 - \frac{137}{29}t - \frac{78}{29} \right]$$

$$= -2.9 \left[ \left( t - \frac{137}{58} \right)^2 - \left( \frac{137}{58} \right)^2 - \frac{78}{29} \right] \text{ (M1) (A1)}$$

$$= -2.9 \left[ \left( t - \frac{137}{58} \right)^2 - 8.269\dots \right]$$

$$= -2.9 \left( t - \frac{137}{58} \right)^2 + 23.98$$

$$= 23.98 - 2.9 \left( t - \frac{137}{58} \right)^2 \text{ (A1)}$$

$$d) \text{ from part (c)} \quad h(t) = 23.98 - 2.9 \left( t - \frac{137}{58} \right)^2$$

maximum  
height

t  
Time when  
max height  
reached

$$\therefore \text{Maximum height} = 23.98$$

$$= 24 \text{ (2 S.F.)}$$

(A1)

$$\text{from part (c)} \Rightarrow t - \frac{137}{58} = 0$$

$$t = \frac{137}{58}$$

$$= 2.36$$

$$= 2.4 \text{ (2 S.F.)}$$

$$\therefore \text{Time when max height reached} = 2.4 \text{ s}$$

(A1)

5. (a) (i) Sketch the graph of  $y = x(x - 2)(x + 1)^2$  stating clearly the points of intersection with the axes

(3)

(ii) State the range of values for which  $f(x) \leq 0$

(1)

(iii) The point with coordinates  $(-2, 0)$  lies on the curve with equation

$$y = (x + k)(x + k - 2)(x + k + 1)^2 \text{ where } k \text{ is a constant.}$$

Find the possible values of  $k$

(1)

(b) On separate axes, sketch the graph  $3y = -x(x - 2)(x + 1)^2$  stating clearly the points of intersection with the axes

$$y = -\frac{1}{3}x(x - 2)(x + 1)^2$$

$\Rightarrow \frac{1}{3}$  amplitude of original curve

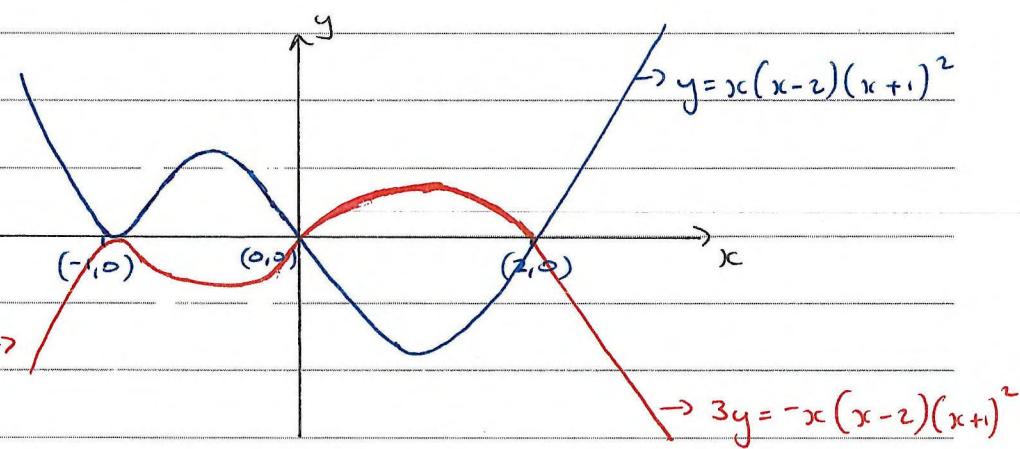
(Total 8 marks)

a, cis  $y = x(x - 2)(x + 1)^2 \rightarrow$  +ve quartic curve  
 $\downarrow \quad \downarrow \quad \downarrow$   
 $x=0 \quad x=2 \quad \text{repeated roots at } x = -1$

(A1) for correct shape

(A2) for 3 correct coordinates

part (b) →



iii,  $f(x) \leq 0$  when  $y$  values are -ve

$\Rightarrow f(x) \leq 0$  when  $0 \leq x \leq 2$

(A1) for correct shape

(A2) for 3 correct coordinates

iv, Original graph

$(-1, 0)$  Horizontal translation  $(-2, 0) \Rightarrow f(x+1) \therefore k=1$   
 1 unit to the left

Transformed Graph

$(0, 0)$  Horizontal translation  $(-2, 0) \Rightarrow f(x+2) \therefore k=2$   
 2 units to the left

$(2, 0)$  Horizontal translation  $(-2, 0) \Rightarrow f(x+4) \therefore k=4$   
 4 units to the left

$\therefore$  Possible values of  $k = 1, 2$  or  $4$

(A1)

6. A has position vector  $5\mathbf{i} - 2\mathbf{j}$  and the point B has position vector  $-4\mathbf{i} + 3\mathbf{j}$ .

Given that C is the point such that  $\overrightarrow{AC} = 2\overrightarrow{AB}$

- (a) find the unit vector in the direction of  $\overrightarrow{OC}$

(3)

D is a point with position vector  $a\mathbf{i} - 3\mathbf{j}$ , where  $a$  is a constant.

Given that  $\overrightarrow{OD} = b\overrightarrow{OA} + \overrightarrow{OB}$ , where  $b$  is a constant.

- (b) Find the values of  $a$  and  $b$

(2)

(Total 5 marks)

$$\text{a), let } \overrightarrow{OC} = \begin{pmatrix} x \\ y \end{pmatrix} \quad \overrightarrow{OA} = \begin{pmatrix} 5 \\ -2 \end{pmatrix} \quad \overrightarrow{OB} = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$$

$$\overrightarrow{AC} = 2 \overrightarrow{AB}$$

$$\overrightarrow{OC} - \overrightarrow{OA} = 2(\overrightarrow{OB} - \overrightarrow{OA}) \quad \text{(M1)}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 5 \\ -2 \end{pmatrix} = 2 \left[ \begin{pmatrix} -4 \\ 3 \end{pmatrix} - \begin{pmatrix} 5 \\ -2 \end{pmatrix} \right]$$

$$\begin{pmatrix} x - 5 \\ y + 2 \end{pmatrix} = 2 \begin{pmatrix} -9 \\ 5 \end{pmatrix}$$

$$\begin{pmatrix} x - 5 \\ y + 2 \end{pmatrix} = \begin{pmatrix} -18 \\ 10 \end{pmatrix}$$

$$\therefore x - 5 = -18 \quad \text{and} \quad y + 2 = 10$$

$$x = -13$$

$$y = 8$$

$$\therefore \overrightarrow{OC} = \begin{pmatrix} -13 \\ 8 \end{pmatrix} \quad \text{(A1)} \quad \text{Unit vector, } \hat{\mathbf{e}} = \frac{\overrightarrow{OC}}{|\overrightarrow{OC}|}$$

$$\text{or } \overrightarrow{OC} = -13\mathbf{i} + 8\mathbf{j}$$

$$= \frac{-13\mathbf{i} + 8\mathbf{j}}{\sqrt{233}}$$

$$|\overrightarrow{OC}| = \sqrt{(-13)^2 + 8^2} \\ = \sqrt{233}$$

$$\therefore \hat{\mathbf{e}} = \frac{1}{\sqrt{233}}(-13\mathbf{i} + 8\mathbf{j}) \quad \text{(A1)}$$

$$b, \quad \vec{OD} = \begin{pmatrix} a \\ -3 \end{pmatrix} \quad \vec{OA} = \begin{pmatrix} 5 \\ -2 \end{pmatrix} \quad \vec{OB} = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$$

$$\vec{OD} = b \vec{OA} + \vec{OB}$$

$$\begin{pmatrix} a \\ -3 \end{pmatrix} = b \begin{pmatrix} 5 \\ -2 \end{pmatrix} + \begin{pmatrix} -4 \\ 3 \end{pmatrix} \quad \text{(M1)}$$

$$\therefore \begin{pmatrix} a \\ -3 \end{pmatrix} = \begin{pmatrix} 5b - 4 \\ -2b + 3 \end{pmatrix}$$

$$\therefore a = 5b - 4 \quad \text{and} \quad -3 = -2b + 3$$

$$-6 = -2b$$

$$\text{Sub } b = 3 \text{ in } a = 5b - 4 \quad b = 3$$

$$a = 5(3) - 4$$

$$a = 11$$

$$\therefore a = 11$$

$$b = 3$$

(A1)

7. (a) Find the first 3 terms of the expansion  $(1 + 3px)^9$  in ascending powers of  $x$  leaving each term in its simplest form (2)

(b) Given that, in the expansion of  $(1 + 3px)^9$ , the coefficient of  $x$  is  $q$  and the coefficient of  $x^2$  is  $4q$ , find the value of  $p$  and  $q$  (3)

(Total 5 marks)

$$\begin{aligned}
 a, \quad (1 + 3px)^9 &= {}^9C_0 1^9 (3px)^0 + {}^9C_1 1^8 (3px)^1 + {}^9C_2 1^7 (3px)^2 + \dots \\
 &= (1)(1)(1) + (9)(1)(3px) + 36(1)(9p^2x^2) + \dots \\
 &= 1 + 27px + 324p^2x^2 + \dots
 \end{aligned}$$

A1

A1

$$\begin{aligned}
 b, \quad \text{Coeff of } x \text{ in expansion is } 27p \\
 \text{Coeff of } x^2 \text{ is } q \\
 \therefore 27p = q \quad \text{--- (1)}
 \end{aligned}$$

Coeff of  $x^2$  in expansion is  $324p^2$

Coeff of  $x^2$  is  $4q$

$$324p^2 = 4q$$

$$81p^2 = q \quad \text{--- (2)}$$

$$\text{Sub (1) in (2)} \Rightarrow 81p^2 = 27p \quad \text{M1}$$

$$81p^2 - 27p = 0$$

$$27p(3p - 1) = 0$$

$$27p = 0 \quad \text{or} \quad 3p - 1 = 0$$

$$p = 0$$

NOT VALID SINCE  
 $p \neq 0$

$$p = \frac{1}{3} \quad \text{A1}$$

$$\begin{aligned}
 \text{Sub } p = \frac{1}{3} \text{ in (1)} \Rightarrow 27\left(\frac{1}{3}\right) = q \\
 q = 9 \quad \text{A1}
 \end{aligned}$$

8. The circle  $C$  has equation  $x^2 + y^2 + 2x - 20y = -51$

The line  $L$  with equation  $3y - 4x = 9$  intersects the circle at the points  $P$  and  $Q$ .  
Given that the  $x$  coordinate of  $Q$  is  $> 0$

(a) Find the centre and radius of the circle

(2)

(b) Find the equation of the tangent at the point  $P$  and the point  $Q$

(4)

Points  $P$  and  $Q$  form the chord  $PQ$  of the circle

(c) Find the equation of the perpendicular bisector of the chord  $PQ$  giving your answers in the form  $ax + by + c = 0$  where  $a$ ,  $b$  and  $c$  are constants

(3)

(d) The perpendicular bisector and the two tangents intersect at a single point.

Find the coordinate of the point of intersection.

(3)

(Total 12 marks)

a, Grouping the  $x$  and  $y$  terms together  $\Rightarrow x^2 + 2x + y^2 - 20y + 51 = 0$

Completing the square  $\Rightarrow (x+1)^2 - 1^2 + (y-10)^2 - 10^2 + 51 = 0$

$$(x+1)^2 + (y-10)^2 - 101 + 51 = 0$$

$$(x+1)^2 + (y-10)^2 = 50$$

$$\therefore \text{Centre} = (-1, 10)$$

A1

$$\therefore \text{Radius} = \sqrt{50}$$

A1

b, Solving the 2 equations simultaneously M1

$$3y - 4x = 9$$

$$3y = 9 + 4x$$

$$y = \frac{4}{3}x + 3$$

Sub  $y = \frac{4}{3}x + 3$  in  $x^2 + y^2 + 2x - 20y + 51 = 0$

$$\Rightarrow x^2 + \left(\frac{4}{3}x + 3\right)^2 + 2x - 20\left(\frac{4}{3}x + 3\right) + 51 = 0$$

$$x^2 + \frac{16x^2 + 8x + 9}{9} + 2x - \frac{80}{3}x - 60 + 51 = 0$$

$$\frac{25x^2}{9} - \frac{50}{3}x = 0$$

$$(x9) \quad 25x^2 - 150x = 0$$

$$25x(x - 6) = 0$$

$$\text{either } 25x = 0 \quad \text{or} \quad x - 6 = 0$$

$$x = 0$$

$$x = 6$$

Finding coordinates of P & Q

Sub  $x=0$  in  $y = \frac{4}{3}x + 3$

$$y = 3$$

Since x-coordinate of Q  $> 0$

$$\therefore P(0, 3)$$

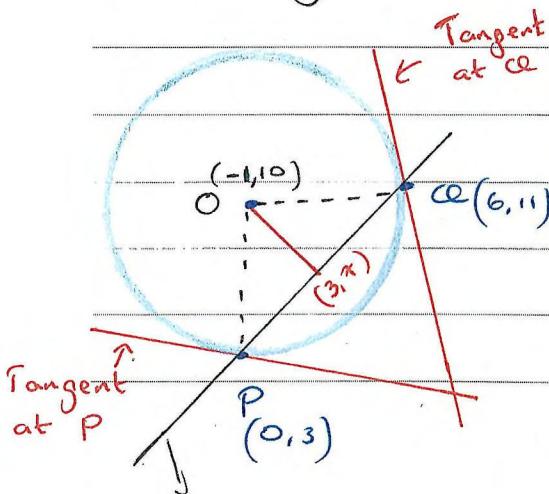
Sub  $x=6$  in  $y = \frac{4}{3}x + 3$

$$\begin{aligned} &= \frac{4}{3}(6) + 3 \\ &= 11 \end{aligned}$$

(A1) For both  
correct coordinate!

$$\therefore Q(6, 11)$$

Finding gradient of tangents at P & Q



$$\text{Gradient of OP} = \frac{10-3}{-1-0} = -7 \Rightarrow$$

$$\text{Gradient of OQ} = \frac{10-11}{-1-6} = \frac{1}{7} \Rightarrow$$

$$\therefore \text{Gradient of tangent at P} = \frac{1}{7}$$

$$\therefore \text{Gradient of tangent at Q} = -7$$

$$3y - 4x = 9$$

Equation of tangent at P

$$\Rightarrow y - 3 = \frac{1}{7}(x - 6)$$

$$y = \frac{1}{7}x + 3 \quad \text{A1}$$

Equation of tangent at Q

$$\Rightarrow y - 11 = -7(x - 6)$$

$$y = -7x + 42 + 11$$

$$y = -7x + 53 \quad \text{A1}$$

c, finding midpoint of PQ  $\Rightarrow \left( \frac{0+6}{2}, \frac{3+11}{2} \right) = (3, 7)$  M1

Finding gradient of b bisection  $\Rightarrow \frac{10-7}{-1-3} = -\frac{3}{4}$   
(from centre of circle (-1, 10) to  
midpt of PQ (3, 7))

.: Equation of b bisection of chord PQ

$$\Rightarrow y - 7 = -\frac{3}{4}(x - 3) \quad \text{M1}$$

$$y = -\frac{3}{4}x + \frac{9}{4} + 7$$

$$y = -\frac{3}{4}x + \frac{37}{4}$$

(x4)

$$4y = -3x + 37$$

$$\therefore 3x + 4y - 37 = 0 \quad \text{A1}$$

d, Equation of perpendicular bisector  $\Rightarrow y = -\frac{3}{4}x + \frac{37}{4}$

Equation of tangent at P  $\Rightarrow y = \frac{1}{7}x + 3$

Solving simultaneously  $\Rightarrow -\frac{3}{4}x + \frac{37}{4} = \frac{1}{7}x + 3$  M1

$$-\frac{3}{4}x - \frac{1}{7}x = 3 - \frac{37}{4}$$

$$\frac{-25}{28}x = \frac{-25}{4}$$

$$x = 7$$

Sub  $x=7$  in  $y = \frac{1}{7}x + 3$

$$\Rightarrow y = \frac{1}{7}(7) + 3$$

$$= 4 \quad \therefore (7, 4) \quad \text{A1}$$

9. (a) Prove that for any positive values of  $a$  and  $b$

$$\frac{4a}{b} + \frac{b}{a} \geq 4 \quad (3)$$

(b) By use of a counter-example, show that this inequality is not true when either  $a$  or  $b$  is not positive. (2)

(Total 5 marks)

a, Using jottings

$$\frac{4a}{b} + \frac{b}{a} \geq 4$$

$$(x_{ab}) \quad \frac{4a^2 + b^2}{ab} \geq 4$$

$$4a^2 + b^2 \geq 4ab$$

$$4a^2 - 4ab + b^2 \geq 0$$

$$(2a-b)(2a-b) \geq 0 \quad \therefore 4a^2 - 4ab + b^2 \equiv (2a-b)^2$$

Now starting proof  $\Rightarrow$  Consider  $(2a-b)^2 \geq 0$

M1

$$4a^2 - 4ab + b^2 \geq 0$$

$$\frac{4a^2 - 4ab + b^2}{ab} \geq 0$$

M1

This step is

possible since

both  $a$  and  $b$

are positive

$$\frac{4a}{b} + \frac{b}{a} \geq 4$$

A1

$\therefore \frac{4a}{b} + \frac{b}{a} \geq 4$  for any positive values of  $a$  and  $b$

b, If  $a = -1$  and  $b = 2$  (M1)

$$\Rightarrow \frac{4(-1)}{2} + \frac{2}{(-1)} = -2 - 2 \quad \left( \text{Sub } a = -1 \text{ and } b = 2 \text{ in } \frac{4a}{b} + \frac{b}{a} \right)$$
$$= -4$$

-4 is not  $\geq 4$

∴ This inequality is not true when either  
a or b is not positive (A1)

10. (a) Show that  $\frac{\sin^4 x + \sin^2 x \cos^2 x}{\cos^2 x - 1} = 1$  (3)

(b) Hence solve the equation  $\frac{\sin^4 x + \sin^2 x \cos^2 x}{\cos^2 x - 1} + 4 = 2\sin^2 x + 3\cos x$  in the interval  $0 \leq x \leq 360^\circ$  (5)

(Total 8 marks)

a) LHS =  $\frac{\sin^4 x + \sin^2 x \cos^2 x}{\cos^2 x - 1}$

M1

$$= \frac{\sin^4 x + \sin^2 x (1 - \sin^2 x)}{\cos^2 x - 1} \quad \left[ \text{Using } \sin^2 x + \cos^2 x = 1 \right]$$

$$= \frac{\sin^4 x + \sin^2 x - \sin^4 x}{\cos^2 x - 1}$$

$$= \frac{\sin^2 x}{-\sin^2 x} \quad \left[ \begin{array}{l} \cos^2 x + \sin^2 x = 1 \\ \cos^2 x - 1 = -\sin^2 x \end{array} \right]$$

$$= -1 \quad \text{A1}$$

$$= \text{RHS}$$

b,  $\frac{\sin^4 x + \sin^2 x \cos^2 x}{\cos^2 x - 1} + 4 = 2\sin^2 x + 3\cos x$

From part (a)  $-1 + 4 = 2\sin^2 x + 3\cos x$  M1

$$3 = 2\sin^2 x + 3\cos x$$

$$\therefore 2\sin^2 x + 3\cos x - 3 = 0$$

$$2(1 - \cos^2 x) + 3\cos x - 3 = 0 \quad \text{M1} \quad \left( \text{Using } \sin^2 x + \cos^2 x = 1 \right)$$

$$2 - 2\cos^2 x + 3\cos x - 3 = 0$$

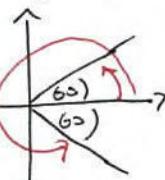
$$-2\cos^2 x + 3\cos x - 1 = 0$$

$$(x-1)(2\cos^2 x - 3\cos x + 1) = 0 \quad \text{A1 A1}$$

$$(2\cos x - 1)(\cos x - 1) = 0$$

$$\cos x = \frac{1}{2} \quad \text{OR} \quad \cos x = 1 \quad \text{M1}$$

$$\therefore x = 0^\circ, 60^\circ, 300^\circ, 360^\circ$$



$$x = 60^\circ, 300^\circ \quad x = 0^\circ, 360^\circ$$

$$\Downarrow (360^\circ - 60^\circ)$$

$$\Downarrow (360^\circ - 0^\circ)$$

11. Solve the following equations, giving your solutions as exact values

(a)  $\ln(3x - 5) = 2 - \ln 3$  (3)

(b)  $e^x + 5e^{-x} = 6$  (3)

(Total 6 marks)

a)  $\ln(3x - 5) = 2 - \ln 3$

$$\ln(3x - 5) + \ln 3 = 2 \quad (\text{Grouping all the } \ln \text{ terms together})$$
$$\ln 3(3x - 5) = 2 \quad (\text{M1}) \quad (\ln a + \ln b = \ln ab)$$
$$3(3x - 5) = e^2 \quad (\text{M1}) \quad (\text{Inverse of } \ln)$$
$$9x - 15 = e^2$$
$$9x = e^2 + 15$$
$$x = \frac{e^2 + 15}{9} \quad (\text{A1})$$

b)  $e^x + 5e^{-x} = 6$

Multiplying by  $e^x$  and creating a quadratic equation  $\Rightarrow e^x \cdot e^x + 5e^{-x} \cdot e^x = 6e^x$  (M1)  
 $e^{2x} + 5e^x = 6e^x$  $e^{2x} - 6e^x + 5 = 0$  (M1)

let  $y = e^x$

$$\therefore y^2 - 6y + 5 = 0$$

$$(y - 5)(y - 1) = 0$$

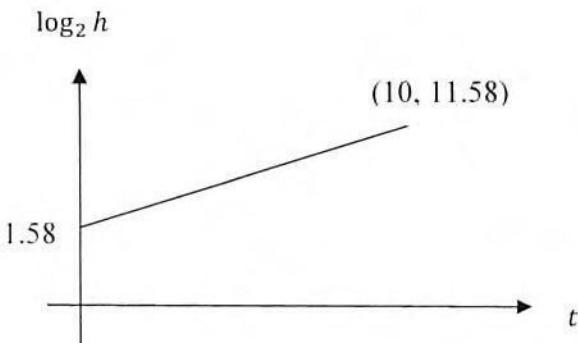
$$y = 5 \quad \text{or} \quad y - 1 = 0$$

(Replacing  $e^x$  back)  $\therefore e^x = 5 \quad \text{or} \quad e^x = 1$

$$x = \ln 5 \quad \text{or} \quad x = 0$$

(A1)

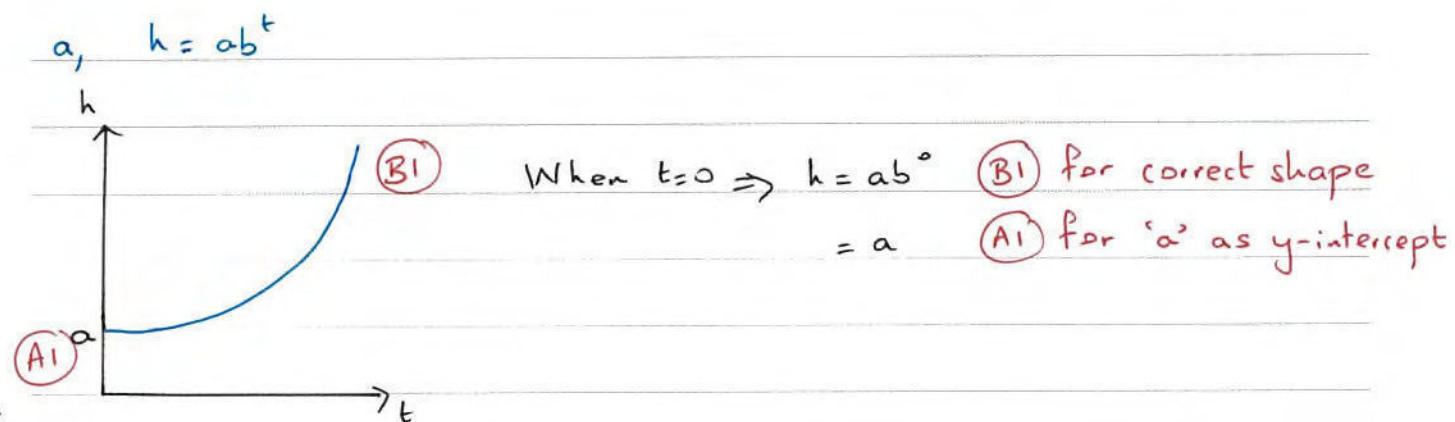
12. A container is being filled with water. After  $t$  seconds, the height  $h$  mm, of the water present in the container can be modelled by the equation  $h = ab^t$ , where  $a$  and  $b$  are constants to be found.



The graph passes through the points  $(0, 1.58)$  and  $(10, 11.58)$

- (a) Sketch the graph of  $h$  against  $t$  (2)
- (b) Comparing the graph in part (a) to the graph of  $\log_2 h$  against  $t$  drawn above, state which graph is more useful for calculations. Explain your reasoning (2)
- (c) Write down an equation of the line (2)
- (d) Find the values of  $a$  and  $b$ , giving your answers to 1 significant figures (4)
- (e) Interpret the meaning of  $a$  in this model (1)
- (f) Suggest one reason why this model is unsuitable for long periods of time and suggest an improvement to the model (2)

(Total 13 marks)



b, The graph of  $\log_2 h$  v/s  $t$  is better for calculations than the graph of  $h$  v/s  $t$  A1

It is easier to find the gradient of a straight line than it is a curve A1

c,  $h = ab^t$

$$\log_2 h = \log_2 ab^t \quad (\text{Take } \log_2 \text{ on both sides})$$

$$\log_2 h = \log_2 a + \log_2 b^t \quad (\text{Applying laws of logarithms})$$

$$\log_2 h = \log_2 a + t \log_2 b \quad \text{(M1)} \quad (\text{Applying laws of logarithms})$$

$$\underbrace{\log_2 h}_y = t \underbrace{\log_2 b}_m + \underbrace{\log_2 a}_c \quad (\text{Rearranging in form } y = mx + c)$$

(0, 1.58) and (10, 11.58)

$$\text{Gradient} = \frac{11.58 - 1.58}{10 - 0} = 1$$

From graph y-intercept = 1.58

$\therefore$  Replacing  $m=1$  and  $c=1.58$  in  $\log_2 h = t \underbrace{\log_2 b}_m + \underbrace{\log_2 a}_c$

$$\Rightarrow \log_2 h = t + 1.58 \quad \text{A1}$$

d, Gradient = 1  $\Rightarrow \log_2 b = 1$

$$b = 2^1$$

$$b = 2 \quad \text{A1}$$

$$Y\text{-intercept} = 1.58 \Rightarrow \log_2 a = 1.58$$

$$a = 2^{1.58}$$

$$= 2.989\dots$$

$$= 3 \quad (1 \text{ s.f.}) \quad \text{A1}$$

$a$ , ' $a$ ' means the initial height of water present  
in the container A1

P, The model implies that the container can  
hold an infinite amount of water which is  
obviously unrealistic A1

An improvement to the model would be to suggest  
that the water in the container reaches its  
maximum height,  $H$  at time  $T$ . Therefore the  
model could be adjusted as when  $t \geq T$ ,  $h = H$

A1

13. Given that  $f(x) = \frac{2}{3}x^3 - 6x^2 + 20x$

(a) Prove that the function  $f(x)$  is increasing for all real values of  $x$  (3)

(b) Sketch the graph of the gradient function  $y = f'(x)$  (2)

(Total 5 marks)

a,  $f(x) = \frac{2}{3}x^3 - 6x^2 + 20x$

$f'(x) = 2x^2 - 12x + 20$  (M1)

Completing the square  $\Rightarrow f'(x) = 2(x^2 - 6x + 10)$

$= 2[(x-3)^2 - (-3)^2 + 10]$  (M1)

$= 2(x-3)^2 + 2$

Since  $(x-3)^2 \geq 0$ ,  $2(x-3)^2 \geq 0$

$\therefore 2(x-3)^2 + 2 \geq 0$  (A1)

for all real values of  $x$

b,  $f'(x) = 2x^2 - 12x + 20$

$$2x^2 - 12x + 20 = 0$$

$\downarrow$        $\downarrow$        $\downarrow$

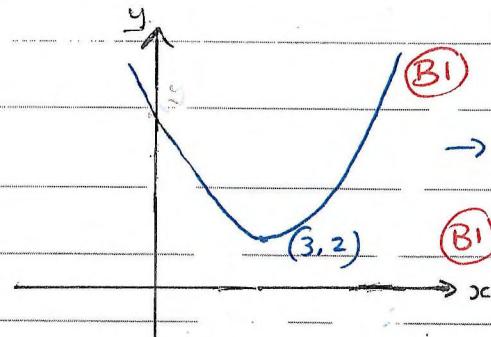
$a$        $b$        $c$

$b^2 - 4ac = (-12)^2 - 4(2)(20)$

$= 144 - 160$

$= -16$

Since  $b^2 - 4ac < 0$ ,  $\therefore$  NO REAL ROOTS



$\rightarrow y = f'(x)$

(B1)

(B1) For correct shape

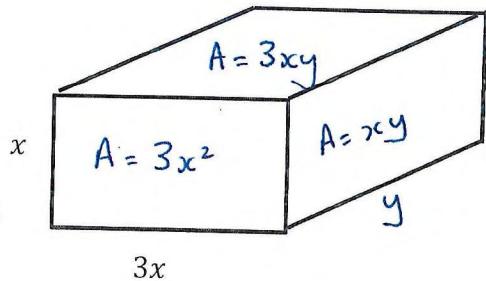
(B1) Turning point in  
correct quadrant

from part (a)  $f'(x) = 2(x-3)^2 + 2$

$\Rightarrow$  coordinates of

turning pt is  $(3, 2)$

14. The diagram below shows a box in the form of a cuboid. The cuboid has a rectangular cross section where the length of the rectangle is equal to three times its width,  $x$  cm. The volume of the cuboid is  $144 \text{ cm}^3$ .



- (a) Show that the surface area  $A$  of the box is  $A = \frac{6}{x} (64 + x^3)$  (3)
- (b) Use calculus to find the minimum value of  $A$  (4)
- (c) Justify that the value of  $A$  you have found is a minimum (2)

(Total 9 marks)

a, let the length of cuboid be  $y$   
 Therefore Volume,  $V = 3x \times y \times x$   
 $V = 3x^2y$

But Volume is  $144 \Rightarrow 3x^2y = 144$   
 $y = \frac{144}{3x^2}$

$y = \frac{48}{x^2}$  — (1)

Surface Area =  $3x^2 + 3x^2 + xy + xy + 3xy + 3xy$   
 $= 6x^2 + 2xy + 6xy$   
 $= 6x^2 + 8xy$  (M1)

$$\text{Surface Area} = 6x^2 + 8xy \quad \text{--- (2)}$$

$$\text{Sub (1) in (2)} \Rightarrow A = 6x^2 + 8x \left( \frac{48}{x^2} \right) \quad (\text{M1})$$

$$= 6x^2 + \frac{384}{x}$$

$$= 6 \left( x^2 + \frac{64}{x} \right) \Leftrightarrow 6 \left( \frac{x^3}{x} + \frac{64}{x} \right)$$

$$= \frac{6}{x} \left( x^3 + 64 \right) \Leftrightarrow \frac{6}{x} (x^3 + 64) \quad (\text{A1})$$

//

b, For minimum value of  $A$ ,  $\frac{dA}{dx} = 0$

$$\therefore \frac{d}{dx} (6x^2 + 384x^{-1}) = 0 \quad \left( \frac{384}{x} = 384x^{-1} \right)$$

$$12x - 384x^{-2} = 0 \quad (\text{M1})$$

$$12x - \frac{384}{x^2} = 0$$

$$(x \neq 0) \quad 12x^3 - 384 = 0$$

$$x^3 = 32$$

$$x = \sqrt[3]{32} \quad (\text{A1})$$

Sub  $x = \sqrt[3]{32}$  in  $A = 6x^2 + \frac{384}{x}$  to find minimum value of  $A$

$$\Rightarrow \text{Minimum value of } A = 6(\sqrt[3]{32})^2 + \frac{384}{\sqrt[3]{32}} \quad (\text{M1})$$

$$= 181.4286\dots$$

$$= 181 \quad (\text{3 s.f.}) \quad (\text{A1})$$

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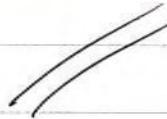
c) finding the 2<sup>nd</sup> derivative of A  $\Rightarrow \frac{d^2A}{dx^2}$

$$\frac{dA}{dx} = 12x - 384x^{-2}$$

$$\therefore \frac{d^2A}{dx^2} = 12 + 768x^{-3} \quad (\text{A1})$$

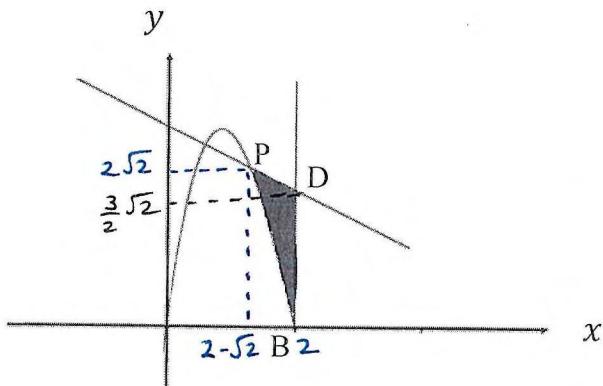
$$\text{Sub } x = \sqrt[3]{32} \Rightarrow \frac{d^2A}{dx^2} = 12 + 768(\sqrt[3]{32})^{-3} \\ = 36$$

Since  $\frac{d^2A}{dx^2} > 0 \therefore A \text{ is a minimum } (\text{A1})$



15. The diagram below shows a sketch of part of the curve  $C$  with equation

$$y = x(x - 2)(x - 4)$$



The point  $P$  lies on  $C$  and has  $x$  coordinate  $2 - \sqrt{2}$ . The curve  $C$  cuts the  $x$ -axis at point  $B$  at  $(2,0)$ . The normal to the curve at  $P$  meets the vertical line at  $B$  at the point  $D$ .

(a) Show that the equation of the normal to the curve at  $P$  is  $2y + x = 2 + 3\sqrt{2}$  (4)

(b) Use calculus to find the exact area of the shaded region (4)

(Total 8 marks)

$$\begin{aligned}
 a, \quad y &= x(x-2)(x-4) \\
 &= (x^2 - 2x)(x-4) \\
 &= x^3 - 4x^2 - 2x^2 + 8x \\
 &= x^3 - 6x^2 + 8x
 \end{aligned}$$

Finding gradient function  $\Rightarrow \frac{dy}{dx} = 3x^2 - 12x + 8$

M1

Finding gradient at P when x-coordinate is  $2-\sqrt{2}$  (M1)

$$\Rightarrow \left. \frac{dy}{dx} \right|_{x=2-\sqrt{2}} = 3(2-\sqrt{2})^2 - 12(2-\sqrt{2}) + 8$$
$$= 18 - 12\sqrt{2} - 24 + 12\sqrt{2} + 8$$
$$= 2$$

∴ Gradient of normal at P is  $-\frac{1}{2}$  (negative reciprocal of 2)

Finding y-coordinate of P ⇒ When  $x = 2-\sqrt{2}$

$$\Rightarrow y = (2-\sqrt{2})^3 - 6(2-\sqrt{2})^2 + 8(2-\sqrt{2})$$
$$= 2\sqrt{2}$$

∴ P  $(2-\sqrt{2}, 2\sqrt{2})$

Finding equation of normal

$$\Rightarrow y - 2\sqrt{2} = -\frac{1}{2} [x - (2-\sqrt{2})] \quad (\text{A1})$$

$$y - 2\sqrt{2} = -\frac{1}{2} (x - 2 + \cancel{\sqrt{2}})$$

careful with change of sign

$$y - 2\sqrt{2} = -\frac{1}{2}x + 1 - \frac{1}{2}\sqrt{2}$$

$$y = \frac{-1}{2}x + \frac{2+3\sqrt{2}}{2}$$

$$(x_2) \quad 2y = -x + 2 + 3\sqrt{2}$$

$$2y + x = 2 + 3\sqrt{2}$$

(A1)

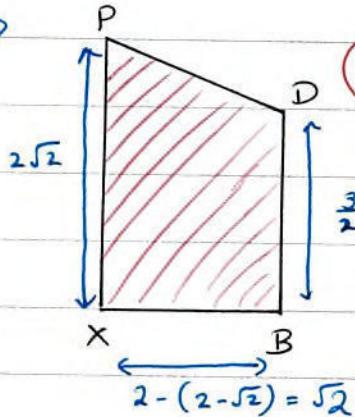
b, Finding coordinate of D  $\Rightarrow$  When  $x=2, 2y = -(2) + 2 + 3\sqrt{2}$   
 $y = \frac{3}{2}\sqrt{2}$

$$\therefore D(2, \frac{3}{2}\sqrt{2})$$

Shaded Area = Area of Trapezium - Area under curve  
 $\times BDP$  from x to B

Area of Trapezium  $\Rightarrow$

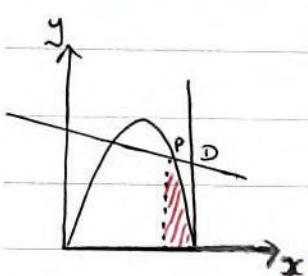
$\times BDP$



(M1)

$$A = \frac{1}{2} (2\sqrt{2} + \frac{3}{2}\sqrt{2}) \times \sqrt{2} = 3.5$$

Area under curve =  $\int_{2-\sqrt{2}}^2 (x^3 - 6x^2 + 8x) dx$  (M1)



$$= \left[ \frac{x^4}{4} - 2x^3 + 4x^2 \right]_{2-\sqrt{2}}^2 \quad (\text{A1})$$

$$= \left[ \frac{(2)^4}{4} - 2(2)^3 + 4(2)^2 \right] - \left[ \frac{(2-\sqrt{2})^4}{4} - 2(2-\sqrt{2})^3 + 4(2-\sqrt{2})^2 \right]$$

$$= 4 - 1$$

$$= 3$$

$\therefore$  Shaded area = Area of Trapezium - Area under curve

$$= \boxed{\text{shaded trapezium}} - \boxed{\text{area under curve}} \\ = 3.5 - 3 = \underline{\underline{0.5}} \quad (\text{A1})$$