Name:........K. NAIKER........

Total Marks:........SOLUTIONS........

GCSE (9-1) Grade 8/9

PROOF

Instructions

- Use **black** ink or ball-point pen.
- **Fill in the boxes** at the top of this page with your name.
- Answer **all** questions.
- Answer the questions in the spaces provided
  - there may be more space than you need.
- **Show all your working out**

Information

- The total mark for this paper is 100.
- The marks for **each** question are shown in brackets.
  - use this as a guide as to how much time to spend on each question.
- Questions labelled with an **asterisk** (*) are ones where the quality of your written communication will be assessed

Advice

- Read each question carefully before you start to answer it
- Attempt every question
- Check your answers if you have time at the end

1. Tanaka says 'When you multiply an odd number and an even number together, you will always get an odd number'.

Show that Tanaka is wrong.

$2 \times 3 = 6$

even × odd = even

∴ Tanaka is wrong

(Total 2 marks)

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2. Tarish says,

'The sum of two prime numbers is always an even number'.

He is **wrong**.
Explain why.

$2 + 3 = 5$

prime prime odd

∴ Tarish is wrong

....

....

(Total 2 marks)

3. The n th even number is $2n$.
The next even number after $2n$ is $2n + 2$.

(a) Explain why.

$2n$ is even, $2n+1$ is odd $\therefore$ $2n+2$ is even

Note → [$2n$ is even and 2 is even, even + even = even]

(1)

(b) Write down an expression, in terms of n, for the next even number after $2n + 2$.

$2n+4$

(1)

(c) Show algebraically that the sum of any 3 consecutive even numbers is always a multiple of 6.

If 1st even number $= 2n$

Next even number $= 2n+2$

Next even number $= 2n+4$

$\therefore \quad 2n + 2n+2 + 2n+4 = 6n+6$

$= 6(n+1)$

which is a multiple of 6

(3)

(Total 5 marks)

4. Here are the first 4 lines of a number pattern.

$1 + 2 + 3 + 4$	=	$(4 \times 3) - (2 \times 1)$
$2 + 3 + 4 + 5$	=	$(5 \times 4) - (3 \times 2)$
$3 + 4 + 5 + 6$	=	$(6 \times 5) - (4 \times 3)$
$4 + 5 + 6 + 7$	=	$(7 \times 6) - (5 \times 4)$

n is the first number in the nth line of the number pattern.

Show that the above number pattern is true for the four consecutive integers n, $(n + 1)$, $(n + 2)$ and $(n + 3)$.

$n + n+1 + n+2 + n+3$

$= 4n + 6$

WHICH IS THE SAME

$(n+3)(n+2) - (n+1)(n)$

$n^2 + 2n + 3n + 6 - (n^2 + n)$

$n^2 + 5n + 6 - n^2 + n$

$4n = 6$

(Total 4 marks)

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5. n is a whole number.

Prove that $n^2 + (n + 1)^2$ is always an odd number.

$n^2 + (n+1)^2 = n^2 + n^2 + 2n + 1$

$= 2n^2 + 2n + 1$

$= 2\underbrace{(n^2 + n)}_{n} + 1$

which is an odd number as $2n+1$ is odd

(Total 2 marks)

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6. n and a are integers.

Explain why $(n^2 - a^2) - (n - a)^2$ is always an integer.

$n^2 - 2an + a^2$

$-1 \times -1 = +1$ $\quad -1 \times +1 = -1$

$$\therefore (n^2 - a^2) - (n-a)^2 = n^2 - a^2 - (n^2 - 2an + a^2)$$
$$= \cancel{n^2} - a^2 - \cancel{n^2} + 2an - a^2$$
$$= -2a^2 + 2an$$
$$= 2a(\underbrace{-a+n}_{\text{integer}})$$

integer [since integer − integer gives integer]

2 × integer = integer

$\therefore 2a(-a+n)$ is an integer

since integer × integer = integer

(Total 2 marks)

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7. n is an integer greater than 1.

Use algebra to show that $(n^2 - 1) + (n - 1)^2$ is always equal to an even number.

$$n^2 - 1 + (n-1)^2 = n^2 - 1 + (n^2 - 2n + 1)$$
$$= n^2 \cancel{-1} + n^2 - 2n \cancel{+1}$$
$$= 2n^2 - 2n$$
$$= 2(\underbrace{n^2 - n}_{m})$$

which is an even number since $2m$ is even

(Total 4 marks)

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8. Prove that the difference between the squares of any two consecutive even numbers is always an odd number multiplied by 4.

let 1st even number = $2n$

Then the consecutive even number = $2n+2$

$$(2n)^2 + (2n+2)^2 = 4n^2 + 4n^2 + 8n + 4$$
$$= 8n^2 + 8n + 4$$
$$= 4(2n^2 + 2n + 1)$$
$$= 4[2\underbrace{(n^2+n)}_{m} + 1]$$
$$= 4(2m+1)$$

$(2m+1)$ is an odd number

$4(2m+1)$ is an odd number multiplied by 4

(Total 4 marks)

9. Prove algebraically that the sum of the squares of two consecutive integers is always an odd number.

let 1st integer = n

Then the next integer = $n+1$

$$n^2 + (n+1)^2 = n^2 + n^2 + 2n + 1$$
$$= 2n^2 + 2n + 1$$
$$= 2\underbrace{(n^2+n)}_{m} + 1$$
$$= 2m+1$$

which is an odd number

(Total 3 marks)

***10.** Prove that the sum of the squares of any two odd numbers is always even.

1st odd = $2n+1$

2nd odd = $2n+3$

$$(2n+1)^2 + (2n+3)^2 = 4n^2 + 4n + 1 + 4n^2 + 12n + 9$$
$$= 8n^2 + 16n + 10$$
$$= 2(\underbrace{4n^2 + 8n + 5}_{m})$$

which is an even as $2m$ is even

(Total 4 marks)

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11. Show that $(n + 3)^2 - (n - 3)^2$ is an even number for all positive integer values of n.

$$(n+3)^2 - (n-3)^2 = n^2 + 6n + 9 - (n^2 - 6n + 9)$$
$$= \cancel{n^2} + 6n + \cancel{9} - \cancel{n^2} + 6n - \cancel{9}$$
$$= 12n$$

If n is positive, then $12n$ is even

(Total 3 marks)

***12.** Prove that

$$(7n + 3)^2 - (7n - 3)^2$$

is a multiple of 12, for all positive integer values of n.

$$(7n+3)^2 - (7n-3)^2 = 49n^2 + 42n + 9 - (49n^2 - 42n + 9)$$
$$= \cancel{49n^2} + 42n \cancel{+ 9} \cancel{- 49n^2} + 42n \cancel{- 9}$$
$$= 84n$$
$$= 12(7n)$$

which is a multiple of 12

(Total 3 marks)

***13.** Prove algebraically that the product of two odd numbers is **always** an odd number.

1st odd $\quad$ 2nd odd

$2n+1 \quad\quad 2n+3$

$$(2n+1)(2n+3) = 4n^2 + 6n + 2n + 3$$
$$= 2(\underbrace{2n^2 + 3n + n}_{m}) + 3$$

which is odd since $2m+3$ is odd

(Total 3 marks)

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14. Prove algebraically that the sum of any two odd numbers is even.

let 1^{st} odd $= 2n+1$

" 2^{nd} odd $= 2n+3$

$$2n+1 + 2n+3 = 4n+4$$

$$= 2(\underbrace{2n+2}_{m})$$

which is even since $2m$ is even

(Total 3 marks)

***15.** Prove algebraically that

$$(2n + 1)^2 - (2n + 1) \quad \text{is an even number}$$

for all positive integer values of n.

$$(2n+1)^2 - (2n+1) = (4n^2 + 4n + 1) - (2n+1)$$

$$= 4n^2 + 4n + 1 - 2n - 1$$

$$= 4n^2 + 2n$$

$$= 2(\underbrace{2n^2 + n}_{m})$$

which is even since $2m$ is even

(Total 3 marks)

*16. Given that a and b are two consecutive even numbers, prove algebraically that

$$\left(\frac{a+b}{2}\right)^2 \text{ is always 1 less than } \frac{a^2+b^2}{2}.$$

let $a = 2n$ and $b = 2n+2$

$$\left(\frac{a+b}{2}\right)^2 = \left(\frac{2n+2n+2}{2}\right)^2$$

$$= \left(\frac{4n+2}{2}\right)^2$$

$$\frac{\cancel{2}(2n+1)}{\cancel{2}}$$

$$= 2n+1$$

$$= (2n+1)^2$$

$$= 4n^2+4n+1$$

which is the same when subtract 1

$$\left(\frac{a^2+b^2}{2}\right) = \frac{(2n)^2+(2n+2)^2}{2}$$

$$= \frac{4n^2+4n^2+8n+4}{2}$$

$$= \frac{8n^2+8n+4}{2}$$

$$= \frac{\cancel{2}(4n^2+4n+2)}{\cancel{2}}$$

$$= 4n^2+4n+2$$

(Total 5 marks)

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17. Prove that $(3n+1)^2 - (3n-1)^2$ is a multiple of 4, for all positive integer values of n.

$$(3n+1)^2 - (3n-1)^2 = (9n^2+6n+1) - (9n^2-6n+1)$$

$$= \cancel{9n^2} + 6n \cancel{+1} \cancel{-9n^2} + 6n \cancel{-1}$$

$$= 12n$$

$$= 4(3n)$$

which is a multiple of 4

(Total 3 marks)

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*18. Prove that the sum of the squares of two consecutive odd numbers is never a multiple of 8.

let 1st odd number = $2n+1$

then the consecutive odd number = $2n+3$

$$(2n+1)^2 + (2n+3)^2 = 4n^2+4n+1 + 4n^2+12n+9$$
$$= 8n^2+16n+10$$
$$= 8(n^2+2n+1)+2$$

which is 2 more than a multiple of 8

$\therefore$ will never be a multiple of 8

(Total 4 marks)

19. Prove that

$(2n + 3)^2 - (2n - 3)^2$ is a multiple of 8

for all positive integer values of n.

$$(2n+3)^2 - (2n-3)^2 = 4n^2+12n+9 - (4n^2-12n+9)$$
$$= \cancel{4n^2}+12n+\cancel{9}-\cancel{4n^2}+12n\cancel{-9}$$
$$= 24n$$
$$= 8(3n)$$

which is a multiple of 8

(Total 3 marks)

20. Prove, using algebra, that the sum of two consecutive whole numbers is always an odd number.

let 1st whole number = n

then the consecutive whole number = $n+1$

$n + n + 1 = 2n + 1$

which is an odd number

(Total 3 marks)

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21. n is an integer

Prove algebraically that the sum of n(n+1) and n+1 is always a square number.

$$n(n+1) + n + 1 = n^2 + n + n + 1$$
$$= n^2 + 2n + 1$$
$$= (n+1)(n+1)$$
$$= (n+1)^2$$

which is a square number

(Total 3 marks)

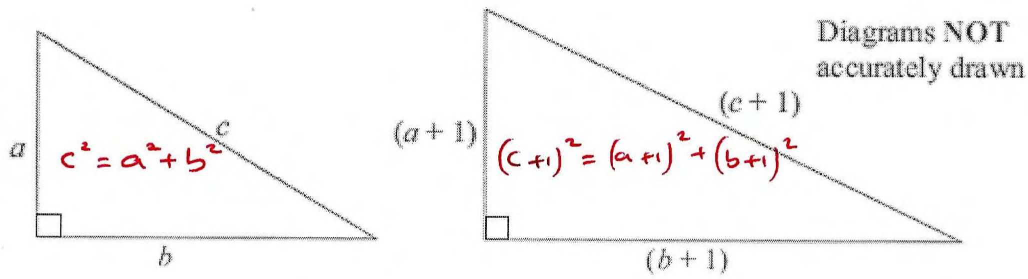
22. Umar thinks $(a+1)^2 = a^2 + 1$ for all values of a.

(a) Show that Umar is wrong.

$(a+1)^2 = (a+1)(a+1) = a^2 + 2a + 1$

(2)

Here are two right-angled triangles.
All the measurements are in centimetres.



(b) Show that $2a + 2b + 1 = 2c$

$(c+1)^2 = (a+1)^2 + (b+1)^2$

$c^2 + 2c + 1 = a^2 + 2a + 1 + b^2 + 2b + 1$

$c^2 + 2c + 1 = a^2 + 2a + b^2 + 2b + 2$

$c^2 + 2c = a^2 + 2a + b^2 + 2b + 1$

$+ = c^2$

Sub $a^2 + b^2 = c^2$ into equation

$\cancel{c^2} + 2c = 2a + \cancel{c^2} + 2b + 1$

$2c = 2a + 2b + 1$

(3)

a, b and c cannot all be integers.

(c) Explain why.

$2c = 2a + 2b + 1$

$c = \dfrac{2a + 2b + 1}{2}$

2a = even

2b = even

$\therefore\ c = \dfrac{\overbrace{\text{even} + \text{even}}^{=\text{even}} + 1}{2}$

$c = \dfrac{\text{odd number}}{2}$

even +1 = odd

= non-integer

If a & b are integers

∴ c will be a non-integer

(1)

(Total 6 marks)

*23. Prove algebraically that the difference between the squares of any two consecutive integers is equal to the sum of these two integers.

let 1st integer = n

Consecutive integer = $n+1$

$n + n + 1 = 2n+1$

$$(n+1)^2 - n^2 = n^2 + 2n + 1 - n^2$$
$$= 2n+1$$

$\therefore$ They are equal

(Total 4 marks)

24. Prove that $(2n + 3)^2 - (2n - 3)^2$ is always a multiple of 12, for all positive integer values of n.

$$(2n+3)^2 - (2n-3)^2 = 4n^2 + 12n + 9 - (4n^2 - 12n + 9)$$
$$= \cancel{4n^2} + 12n \cancel{+9} \cancel{-4n^2} + 12n \cancel{-9}$$
$$= 24n$$
$$= 12(2n)$$

which is a multiple of 12

(Total 3 marks)

25. Prove that the sum of 3 consecutive odd numbers is always a multiple of 3

let 1st odd number = $2n + 1$

2nd consecutive odd = $2n+3$

3rd consecutive odd = $2n+5$

$2n+1 + 2n+3 + 2n+5 = 6n + 9$

$= 3(2n+3)$

which is a multiple of 3

(Total 3 marks)

26. Prove algebraically that the sum of the squares of any 2 even positive integers is always a multiple of 4

$2n$ & $4n$

$(2n)^2 + (4n)^2 = 4n^2 + 16n^2$

$= 20n^2$

$= 4(5n^2)$

∴ which is a multiple of 4

(Total 3 marks)

27. Prove that the sum of the squares of 2 consecutive odd numbers is always 2 more than a multiple of 8

let 1^{st} odd number $= 2n+1$

then consecutive odd $= 2n+3$

$$(2n+1)^2 + (2n+3)^2 = 4n^2 + 4n + 1 + 4n^2 + 16n + 9$$
$$= 8n^2 + 16n + \underbrace{10}_{8+2}$$
$$= 8(n^2 + 2n + 1) + 2$$

which is 2 more than a multiple of 8

(Total 3 marks)

28. Prove algebraically that the sums of the squares of any 2 consecutive even number is always 4 more than a multiple of 8

2n and 2n + 2

$$(2n)^2 + (2n+2)^2 = 4n^2 + 4n^2 + 8n + 4$$
$$= 8n^2 + 8n + 4$$
$$= 8(n^2 + n) + 4$$

which is 4 more than a multiple of 8

(Total 3 marks)

$n(n+1) \quad + \quad (n+1)$

29. The product of 2 consecutive positive integers is added to the larger of the two integers.

Prove that the result is always a square number

let 1^{st} integer $= n$

2^{nd} integer $= n+1$

$$n(n+1) + n+1 = n^2+n+n+1$$

$$= n^2+2n+1$$

$$=(n+1)(n+1)$$

$$=(n+1)^2$$

which is a square number

(Total 3 marks)

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30. c is a postive integer

Prove that $\dfrac{6c^3+30c}{3c^2+15}$ is an even number

Factorise both numerator and denominator $\Rightarrow$ $\dfrac{6c\,\cancel{(c^2+5)}}{3\,\cancel{(c^2+5)}} = \dfrac{6c}{3}$

$$= 2c$$

which is an even number since it is a multiple of 2

(Total 4 marks)

TOTAL FOR PAPER IS 100 MARKS