C3 PROOF — Answers - Worksheet A

1 **a** e.g. $a = -2, b = 1$ $\Rightarrow$ $a^2 - b^2 = 4 - 1 = 3$ $\Rightarrow$ $a^2 - b^2 > 0$
and $a - b = -2 - 1 = -3$ $\Rightarrow$ $a - b < 0$
[any negative value of a such that $|\,a\,| > |\,b\,|$]

b 7 7 is prime and divisible by 7 [no other examples]

c e.g. $x = \sqrt{2}$, $y = 2\sqrt{2}$ $\Rightarrow$ x and y irrational
and $xy = 4$ which is rational [many other examples]

d e.g. $x = -90$ $\Rightarrow$ $\cos(90 - |\,x\,|)° = \cos 0 = 1$
and $\sin x° = \sin(-90°) = -1$ [any –ve x except multiples of 180]

2 **a** true any number divisible by 6 is also divisible by 2 and $\therefore$ not prime

b

n	1	2	3	4	5
$3^n + 2$	5	11	29	83	245

false e.g. $n = 5$ $\Rightarrow$ $3^n + 2 = 245$ which is divisible by 5 and $\therefore$ not prime
[many other examples]

c false e.g. $n = 4$ $\Rightarrow$ $\sqrt{n} = 2$ which is rational [many other examples]

d true b divisible by c $\Rightarrow$ $b = kc,\ k \in \mathbb{Z}$
a divisible by b $\Rightarrow$ $a = lb,\ l \in \mathbb{Z}$ $\Rightarrow$ $a = klc$ $\therefore$ a is divisible by c

3 **a** assume n^3 odd and n even, where $n \in \mathbb{Z}^+$
n even $\Rightarrow$ $n = 2m,\ m \in \mathbb{Z}$
$\Rightarrow$ $n^3 = (2m)^3 = 8m^3 = 2(4m^3)$
$4m^3 \in \mathbb{Z}$ $\therefore$ n^3 even
$\Rightarrow$ contradiction $\therefore$ n odd

b assume x irrational and $\sqrt{x}$ rational
$\sqrt{x}$ rational $\Rightarrow$ $\sqrt{x} = \frac{p}{q}$, $p, q \in \mathbb{Z}$
$\Rightarrow$ $x = \frac{p^2}{q^2}$, $p^2, q^2 \in \mathbb{Z}$ $\therefore$ x rational
$\Rightarrow$ contradiction $\therefore$ $\sqrt{x}$ irrational

c assume bc not divisible by a and b divisible by a where $a, b, c \in \mathbb{Z}$
b divisible by a $\Rightarrow$ $b = ka,\ k \in \mathbb{Z}$
$\Rightarrow$ $bc = kac$ which is divisible by a
$\Rightarrow$ contradiction $\therefore$ b is not divisible by a

d assume $n^2 - 4n$ odd and n even, where $n \in \mathbb{Z}^+$
n even $\Rightarrow$ $n = 2m,\ m \in \mathbb{Z}$
$\Rightarrow$ $n^2 - 4n = (2m)^2 - 4(2m) = 4m^2 - 8m = 2(2m^2 - 4m)$
$2m^2 - 4m \in \mathbb{Z}$ $\therefore$ $n^2 - 4n$ even
$\Rightarrow$ contradiction $\therefore$ n odd

e assume $m^2 - n^2 = 6$, where $m, n \in \mathbb{Z}^+$
$m^2 - n^2 = 6$ $\Rightarrow$ $(m + n)(m - n) = 6$
$m, n \in \mathbb{Z}^+$ $\Rightarrow$ $(m + n), (m - n) \in \mathbb{Z}$, $(m + n) > (m - n)$ and $(m + n) > 0$
$\therefore$ $m + n = 6$ and $m - n = 1$ or $m + n = 3$ and $m - n = 2$
adding $\Rightarrow$ $2m = 7$ or $2m = 5$
$\Rightarrow$ $m = \frac{7}{2}$ or $m = \frac{5}{2}$ $\Rightarrow$ m not an integer
$\Rightarrow$ contradiction $\therefore$ no positive integer solutions

4 **a** assume $x^2 + y^2$ divisible by 4 and x, y odd integers

x, y odd $\Rightarrow$ $x = 2m + 1,\ m \in \mathbb{Z}$ and $y = 2n + 1,\ n \in \mathbb{Z}$

$\Rightarrow$ $x^2 + y^2 = (2m + 1)^2 + (2n + 1)^2$

$= 4m^2 + 4m + 1 + 4n^2 + 4n + 1$

$= 4(m^2 + m + n^2 + n) + 2$

$m^2 + m + n^2 + n \in \mathbb{Z}$ $\therefore\ x^2 + y^2$ not divisible by 4

$\Rightarrow$ contradiction $\therefore$ x and y not both odd

b assume $x^2 + y^2$ divisible by 4, x odd integer and y even integer

x odd, y even $\Rightarrow$ $x = 2m + 1,\ m \in \mathbb{Z}$ and $y = 2n,\ n \in \mathbb{Z}$

$\Rightarrow$ $x^2 + y^2 = (2m + 1)^2 + (2n)^2$

$= 4m^2 + 4m + 1 + 4n^2$

$= 4(m^2 + m + n^2) + 1$

$m^2 + m + n^2 \in \mathbb{Z}$ $\therefore\ x^2 + y^2$ not divisible by 4

$\Rightarrow$ contradiction $\therefore$ x odd and y even not possible

same argument applies with x even and y odd

part **a** shows x and y can't both be odd

$\therefore$ x and y both even

5 **a** false e.g. $a = 2, b = 4$ $\Rightarrow$ $\log_a b = 2$ which is rational

[many other examples]

b true $(2n + 1)$ and $(2n + 3)$, $n \in \mathbb{Z}$ represent any two consecutive odd integers

$(2n + 3)^2 - (2n + 1)^2 = 4n^2 + 12n + 9 - (4n^2 + 4n + 1)$

$= 8n + 8$

$= 8(n + 1)$

$n + 1 \in \mathbb{Z}$ $\therefore$ difference is divisible by 8

c false e.g. $n = 13$ $\Rightarrow$ $n^2 + 3n + 13 = 13(13 + 3 + 1)$ which is divisible by 13

[many other examples]

d true $x^2 - 2y(x - y) = x^2 - 2xy + 2y^2$

$= x^2 - 2xy + y^2 + y^2$

$= (x - y)^2 + y^2$

for real x and y, $(x - y)^2 \geq 0$ and $y^2 \geq 0$ $\therefore\ x^2 - 2y(x - y) \geq 0$

6 **a** $\sqrt{2} = \frac{p}{q},\ p, q \in \mathbb{Z}$ $\Rightarrow$ $2 = \frac{p^2}{q^2}$ $\Rightarrow$ $p^2 = 2q^2$

$\Rightarrow$ p^2 even $\Rightarrow$ p even

b assume $\sqrt{2}$ rational $\Rightarrow$ $\sqrt{2} = \frac{p}{q}$, $p, q \in \mathbb{Z}$ and p, q co-prime

part **a** $\Rightarrow$ p even $\Rightarrow$ $p = 2n,\ n \in \mathbb{Z}$

$\Rightarrow$ $(2n)^2 = 2q^2$

$\Rightarrow$ $q^2 = 2n^2$

$\Rightarrow$ q^2 even $\Rightarrow$ q even

$\Rightarrow$ p and q both even $\therefore$ not co-prime

$\Rightarrow$ contradiction $\therefore$ $\sqrt{2}$ is irrational

C3 PROOF — Answers - Worksheet B

1 **a** e.g. $x = \frac{1}{8}$ $\Rightarrow$ $\sqrt[3]{x} = \frac{1}{2}$, $\frac{1}{2} > \frac{1}{8}$

[any value of x in the interval $0 < x < 1$]

b e.g. $n = 7$ $\Rightarrow$ $n^3 - n + 7 = 7(49 - 1 + 1)$ which is divisible by 7

[many other examples]

2 assume $\sqrt{\pi}$ is rational $\Rightarrow$ $\sqrt{\pi} = \frac{p}{q}$, $p, q \in \mathbb{Z}$

$\Rightarrow$ $\pi = \frac{p^2}{q^2}$, $p^2, q^2 \in \mathbb{Z}$ $\therefore$ π rational

$\Rightarrow$ contradiction $\therefore$ $\sqrt{\pi}$ irrational

3 consider $15x^2 - 11x + 2 < 0$

$\Rightarrow$ $(5x - 2)(3x - 1) < 0$

$\Rightarrow$ $\frac{1}{3} < x < \frac{2}{5}$

e.g. $x = 0.35$ $\Rightarrow$ $15x^2 - 11x + 2 = -0.0125$, $-0.0125 < 0$

[any value of x in the interval $\frac{1}{3} < x < \frac{2}{5}$]

4 **a** $n^2 + 2n = (2m + 1)^2 + 2(2m + 1)$

$= 4m^2 + 4m + 1 + 4m + 2$

$= 4m^2 + 8m + 3$

b assume $n^2 + 2n$ even and n odd, where $n \in \mathbb{Z}$

n odd $\Rightarrow$ $n = 2m + 1$, $m \in \mathbb{Z}$

$\Rightarrow$ $n^2 + 2n = 4m^2 + 8m + 3 = 2(2m^2 + 4m + 1) + 1$

$2m^2 + 4m + 1 \in \mathbb{Z}$ $\therefore$ $n^2 + 2n$ odd

$\Rightarrow$ contradiction $\therefore$ n even

5 **a** $k \cos x - \text{cosec } x = 0$ $\Rightarrow$ $k \cos x = \frac{1}{\sin x}$

$\Rightarrow$ $k \sin x \cos x = 1$

$\Rightarrow$ $\frac{1}{2} k \sin 2x = 1$

$\Rightarrow$ $\sin 2x = \frac{2}{k}$

$|\sin 2x| \le 1$ $\Rightarrow$ $\left|\frac{2}{k}\right| \le 1$

$\Rightarrow$ $|k| \ge 2$

b $3 \cos x - \text{cosec } x = 0$ $\Rightarrow$ $\sin 2x = \frac{2}{3}$

$2x = 41.810, 180 - 41.810, 360 + 41.810, 540 - 41.810$

$2x = 41.810, 138.190, 401.810, 498.190$

$x = 20.9, 69.1, 200.9, 249.1$ (1dp)

6 assume $x^2 - y^2 = 1$, where $x, y \in \mathbb{Z}^+$

$x^2 - y^2 = 1 \Rightarrow (x + y)(x - y) = 1$

$x, y \in \mathbb{Z}^+ \Rightarrow (x + y), (x - y) \in \mathbb{Z}$ and $(x + y) > 0$

$\therefore\ x + y = 1$ and $x - y = 1$

adding $\Rightarrow 2x = 2$

$\Rightarrow x = 1$

$\Rightarrow y = 0$

$\Rightarrow$ contradiction $\therefore$ no positive integer solutions

7 **a** false e.g. $a = \sqrt{2}$, $b = 1 - \sqrt{2}$ $\Rightarrow$ a and b irrational

and $a + b = 1$ which is rational

[many other examples]

b true m, n consecutive odd integers $\Rightarrow m = 2a + 1, n = 2a + 3,\ a \in \mathbb{Z}$

$\Rightarrow m + n = 2a + 1 + 2a + 3 = 4a + 4 = 4(a + 1)$

$a + 1 \in \mathbb{Z}\ \therefore\ m + n$ divisible by 4

c false e.g. $x = \frac{5\pi}{3} \Rightarrow \cos x = \frac{1}{2}$ and $1 + \sin x = 1 - \frac{\sqrt{3}}{2}$, $\frac{1}{2} > 1 - \frac{\sqrt{3}}{2}$

[any value of x of the form $2n\pi + y,\ n \in \mathbb{Z},\ -\frac{\pi}{2} < y < 0$]

8 **a** $\log_2 3 = \frac{p}{q} \Rightarrow 2^{\frac{p}{q}} = 3$

$\Rightarrow (2^{\frac{p}{q}})^q = 3^q$

$\Rightarrow 2^p = 3^q$

b assume $\log_2 3$ is rational $\Rightarrow \log_2 3 = \frac{p}{q},\ p, q \in \mathbb{Z},\ q \neq 0$

$\Rightarrow 2^p = 3^q$

2 and 3 are co-prime $\Rightarrow p = q = 0$

$\Rightarrow$ contradiction $\therefore \log_2 3$ is irrational

c e.g. $a = 2, b = \sqrt{2}$ $\Rightarrow$ a rational and b irrational

and $\log_a b = \frac{1}{2}$ which is rational

[many other examples]

9 **a** $y = \frac{x-2}{4x}$ swap $x = \frac{y-2}{4y}$

$4xy = y - 2$

$y(4x - 1) = -2$

$y = \frac{2}{1-4x}$

$f^{-1}(x) = \frac{2}{1-4x},\ x \in \mathbb{R},\ x \neq \frac{1}{4}$

b $f(x) = f^{-1}(x) \Rightarrow \frac{x-2}{4x} = \frac{2}{1-4x}$

$\Rightarrow (x - 2)(1 - 4x) = 8x$

$\Rightarrow 4x^2 - x + 2 = 0$

$b^2 - 4ac = 1 - 32 = -31$

$b^2 - 4ac < 0 \Rightarrow$ no real roots

$\therefore$ no real values of x for which $f(x) = f^{-1}(x)$