C3 PROOF — Worksheet A

1 Give a counter-example to prove that each of the following statements is false.

a If $a^2 - b^2 > 0$, where a and b are real, then $a - b > 0$.

b There are no prime numbers divisible by 7.

c If x and y are irrational and $x \neq y$, then xy is irrational.

d For all real values of x, $\cos(90 - |x|)° = \sin x°$.

2 For each statement, either prove that it is true or find a counter-example to prove that it is false.

a There are no prime numbers divisible by 6.

b $(3^n + 2)$ is prime for all positive integer values of n.

c $\sqrt{n}$ is irrational for all positive integers n.

d If a, b and c are integers such that a is divisible by b and b is divisible by c, then a is divisible by c.

3 Use proof by contradiction to prove each of the following statements.

a If n^3 is odd, where n is a positive integer, then n is odd.

b If x is irrational, then $\sqrt{x}$ is irrational.

c If a, b and c are integers and bc is not divisible by a, then b is not divisible by a.

d If $(n^2 - 4n)$ is odd, where n is a positive integer, then n is odd.

e There are no positive integers, m and n, such that $m^2 - n^2 = 6$.

4 Given that x and y are integers and that $(x^2 + y^2)$ is divisible by 4, use proof by contradiction to prove that

a x and y are not both odd,

b x and y are both even.

5 For each statement, either prove that it is true or find a counter-example to prove that it is false.

a If a and b are positive integers and $a \neq b$, then $\log_a b$ is irrational.

b The difference between the squares of any two consecutive odd integers is divisible by 8.

c $(n^2 + 3n + 13)$ is prime for all positive integer values of n.

d For all real values of x and y, $x^2 - 2y(x - y) \geq 0$.

6 **a** Prove that if

$$\sqrt{2} = \frac{p}{q},$$

where p and q are integers, then p must be even.

b Use proof by contradiction to prove that $\sqrt{2}$ is irrational.

C3 PROOF — Worksheet B

1 Prove, by counter-example, that each of the following statements is false.

a For all positive real values of x, $\sqrt[3]{x} \leq x$. **(2)**

b For all positive integer values of n, $(n^3 - n + 7)$ is prime. **(2)**

2 Use proof by contradiction to prove that $\sqrt{\pi}$ is irrational.

(You may assume that π is irrational). **(4)**

3 Find a counter-example to prove that the statement

"$15x^2 - 11x + 2 \geq 0$ for all real values of x"

is false. **(4)**

4 **a** Given that $n = 2m + 1$, find and simplify an expression in terms of m for $n^2 + 2n$. **(1)**

b Hence, use proof by contradiction to prove that if $(n^2 + 2n)$ is even, where n is an integer, then n is even. **(5)**

5 **a** Prove that if the equation

$$k\cos x - \operatorname{cosec} x = 0,$$

where k is a constant, has real solutions, then $|k| \geq 2$. **(5)**

b Find the values of x in the interval $0 \leq x \leq 360$ for which

$$3\cos x° - \operatorname{cosec} x° = 0.$$ **(3)**

6 Use proof by contradiction to prove that there are no positive integers, x and y, such that

$$x^2 - y^2 = 1.$$ **(6)**

7 For each statement, either prove that it is true or find a counter-example to prove that it is false.

a If a and b are irrational and $a \neq b$, then $(a + b)$ is irrational. **(2)**

b If m and n are consecutive odd integers, then $(m + n)$ is divisible by 4. **(3)**

c For all real values of x, $\cos x \leq 1 + \sin x$. **(2)**

8 **a** Show that if $\log_2 3 = \frac{p}{q}$, then

$$2^p = 3^q.$$ **(2)**

b Use proof by contradiction to prove that $\log_2 3$ is irrational. **(4)**

c Prove, by counter-example, that the statement

"if a is rational and b is irrational then $\log_a b$ is irrational"

is false. **(2)**

9 The function f is defined by

$$\mathrm{f} : x \rightarrow \frac{x-2}{4x}, \quad x \in \mathbb{R}, \quad x \neq 0.$$

a Find an expression for the inverse function, $\mathrm{f}^{-1}(x)$, and state its domain. **(5)**

b Prove that there are no real values of x for which

$$\mathrm{f}(x) = \mathrm{f}^{-1}(x).$$ **(4)**